

ON DEGENERATE q -EULER POLYNOMIALSDMITRY V. DOLGY¹, TAEKYUN KIM², JIN-WOO PARK^{3,*}, AND JONG-JIN SEO⁴

ABSTRACT. Recently, Kim introduced Carlitz's type q -Euler numbers and polynomials (see [5]). In this paper, we construct the degenerate Carlitz's type q -Euler numbers and polynomials which are derived from the fermionic p -adic q -integral on \mathbb{Z}_p . Finally, we give some identities and properties of these numbers and polynomials

1. INTRODUCTION

Let p be a fixed odd prime number. Throughout this paper, \mathbb{Z}_p , \mathbb{Q}_p and \mathbb{C}_p will, respectively, denote the ring of p -adic rational integers, the field of p -adic rational numbers and the completion of the algebraic closure of \mathbb{Q}_p . Let ν_p be the normalized exponential valuation of \mathbb{C}_p with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$. Let q be an indeterminate in \mathbb{C}_p such that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$. The q -analogue of x is defined as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q \rightarrow 1} x = x$.

As is well known, the *Euler polynomials* are defined by the generating function to be

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (\text{see [1-11]}). \quad (1.1)$$

When $x = 0$, $E_n = E_n(0)$ are called the *Euler numbers*. From (1.1), we can derive the recurrence relation for the Euler numbers as follows:

$$(E + 1)^n + E_n = 2\delta_{0,n}, \quad (n \geq 1), \quad (1.2)$$

with the usual convention about replacing E^n by E_n (see [1-12]).

In [5, 6], Kim considered Carlitz's type q -Euler numbers as follows:

$$E_{0,q} = 1, \quad q(qE_q + 1)^n + E_{n,q} = 0, \quad (n \geq 1), \quad (1.3)$$

with the usual convention about replacing E_q^n by $E_{n,q}$.

The q -Euler polynomials are defined as

$$E_{n,q}(x) = \sum_{l=0}^n \binom{n}{l} q^{lx} E_{l,q} [x]_q^{n-l}, \quad (n \geq 0), \quad (\text{see [5, 6]}). \quad (1.4)$$

Let $C(\mathbb{Z}_p)$ be the space of all continuous functions on \mathbb{Z}_p . For $f \in C(\mathbb{Z}_p)$, the *fermionic p -adic q -integral* on \mathbb{Z}_p is defined by Kim to be

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x) (-q)^x, \quad (1.5)$$

where $[x]_q = \frac{1-(-q)^x}{1+q}$, (see [5, 6]).

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From (1.5), we have

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \text{ where } f_1(x) = f(x+1).$$

The q -Euler polynomials can be represented by the fermionic p -adic q -integral on \mathbb{Z}_p as follows:

$$\sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_{-q}(y), \text{ (see [5, 6]).} \quad (1.6)$$

Thus, by (1.6), we get

$$\int_{\mathbb{Z}_p} [x+y]_q^n d\mu_{-q}(y) = E_{n,q}(x), \text{ } (n \geq 0). \quad (1.7)$$

In this paper, we consider the degenerate Carlitz q -Euler polynomials and numbers which are derived from the fermionic p -adic q -integral on \mathbb{Z}_p and we investigate some properties and identities of those polynomials.

2. DEGENERATE CARLITZ q -BERNOULLI NUMBERS AND POLYNOMIALS

In this section, we assume that $\lambda, t \in \mathbb{C}_p$ with $|\lambda t|_p < p^{-\frac{1}{p-1}}$ so that $(1 + \lambda t)^{\frac{x}{\lambda}} = \exp\left(\frac{x}{\lambda} \log(1 + \lambda t)\right)$.

It is well known that

$$\lim_{\lambda \rightarrow 0} (1 + \lambda t)^{\frac{1}{\lambda}} = \lim_{\lambda \rightarrow 0} \sum_{n=0}^{\infty} \left(\frac{1}{\lambda}\right) \lambda^n t^n = \sum_{n=0}^{\infty} \frac{t^n}{n!} = e^{xt}. \quad (2.1)$$

From (1.6) and (2.1), we consider the *degenerate q -Euler polynomials* which are given by the generating function to be

$$\int_{\mathbb{Z}_p} (1 + \lambda t)^{\frac{1}{\lambda} [x+y]_q} d\mu_{-q}(y) = \sum_{n=0}^{\infty} E_{n,q}(x|\lambda) \frac{t^n}{n!}. \quad (2.2)$$

Note that

$$\begin{aligned} \sum_{n=0}^{\infty} \lim_{\lambda \rightarrow 0} E_{n,q}(x|\lambda) \frac{t^n}{n!} &= \lim_{\lambda \rightarrow 0} \int_{\mathbb{Z}_p} (1 + \lambda t)^{\frac{[x+y]_q}{\lambda}} d\mu_{-q}(y) \\ &= \int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_{-q}(y) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \end{aligned} \quad (2.3)$$

Thus, by (2.3), we get

$$\lim_{\lambda \rightarrow 0} E_{n,q}(x|\lambda) = E_{n,q}(x), \text{ } (n \geq 0). \quad (2.4)$$

When $x = 0$, $E_{n,q}(\lambda) = E_{n,q}(0|\lambda)$ are called *degenerate q -Euler numbers*.

From (2.2), we have

$$\begin{aligned} \sum_{n=0}^{\infty} E_{n,q}(x|\lambda) \frac{t^n}{n!} &= \sum_{m=0}^{\infty} \lambda^{-m} \int_{\mathbb{Z}_p} [x+y]_q^m d\mu_{-q}(y) \frac{(\log(1+t))^m}{m!} \\ &= \sum_{m=0}^{\infty} \lambda^{-m} E_{m,q}(x) \sum_{n=m}^{\infty} S_1(n, m) \frac{\lambda^n t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \lambda^{n-m} E_{m,q}(x) S_1(n, m) \right) \frac{t^n}{n!}. \end{aligned} \quad (2.5)$$

By comparing the coefficients on the both sides of (2.5), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$E_{n,q}(x|\lambda) = \sum_{m=0}^n \lambda^{n-m} E_{m,q}(x) S_1(n, m),$$

where $S_1(n, m)$ is the Stirling number of the first kind.

Replacing t by $\frac{1}{\lambda} (e^{\lambda t} - 1)$ in (2.2), we get

$$\begin{aligned} \sum_{m=0}^{\infty} E_{m,q}(x|\lambda) \lambda^{-m} \frac{(e^{\lambda t} - 1)^m}{m!} &= \int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_{-q}(y) \\ &= \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \end{aligned} \quad (2.6)$$

On the other hand,

$$\begin{aligned} \sum_{m=0}^{\infty} E_{m,q}(x|\lambda) \lambda^{-m} \frac{1}{m!} (e^{\lambda t} - 1)^m &= \sum_{m=0}^{\infty} E_{m,q}(x|\lambda) \lambda^{-m} \sum_{n=m}^{\infty} S_2(n, m) \frac{\lambda^n t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n E_{m,q}(x|\lambda) \lambda^{n-m} S_2(n, m) \right) \frac{t^n}{n!}. \end{aligned} \quad (2.7)$$

Therefore, by (2.6) and (2.7), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$E_{n,q}(x) = \sum_{m=0}^n E_{m,q}(x|\lambda) \lambda^{n-m} S_2(n, m).$$

where $S_2(n, m)$ is the Stirling number of the second kind.

We observe that

$$(1 + \lambda t)^{\frac{[x+y]_q}{\lambda}} = \sum_{n=0}^{\infty} \left(\frac{[x+y]_q}{\lambda} \right)_n \frac{\lambda^n t^n}{n!} \quad (2.8)$$

where $(x)_n = x(x-1) \cdots (x-n+1)$.

Now we define the *degenerate falling factorial* as follows:

$$(x)_{n,\lambda} = x(x-\lambda)(x-2\lambda) \cdots (x-(n-1)\lambda). \quad (2.9)$$

Note that $(x)_{n,1} = x(x-1) \cdots (x-n+1) = (x)_n$, $n \geq 0$. Therefore, by (2.2) and (2.9), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, we have*

$$\int_{\mathbb{Z}_p} ([x+y]_q)_{n,\lambda} d\mu_{-q}(y) = E_{n,q}(x|\lambda),$$

where $([x+y]_q)_{n,\lambda} = [x+y]_q([x+y]_q - \lambda) \cdots ([x+y]_q - (n-1)\lambda)$.

From (1.7), we can easily derive the following equation:

$$\begin{aligned} \int_{\mathbb{Z}_p} [x+y]_q^n d\mu_{-q}(y) &= \frac{[2]_q}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lx} \frac{1}{1+q^{l+1}} \\ &= [2]_q \sum_{m=0}^{\infty} (-1)^m [x+m+1]_q^n. \end{aligned} \quad (2.10)$$

Thus, by (2.10), we get the generating function of the q -Euler polynomials as follows:

$$\sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (1-)^m e^{[x+m+1]_q}. \quad (2.11)$$

From (2.2) and (2.11), we can derive the generating function of degenerate q -Euler polynomials which is given by

$$\sum_{n=0}^{\infty} E_{n,q}(x|\lambda) \frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^m (1+\lambda t)^{\frac{[x+m+1]_q}{\lambda}}. \quad (2.12)$$

By (2.12), we easily get

$$\sum_{n=0}^{\infty} E_{n,q}(x|\lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left([2]_q \sum_{m=0}^{\infty} (-1)^m ([x+m+1]_q)_{n,\lambda} \right) \frac{t^n}{n!}. \quad (2.13)$$

Therefore, by (2.13), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$E_{n,q}(x|\lambda) = [2]_q \sum_{m=0}^{\infty} (-1)^m ([x+m+1]_q)_{n,\lambda}.$$

We observe that

$$\begin{aligned} (1+\lambda t)^{\frac{[x+y]_q}{\lambda}} &= (1+\lambda t)^{\frac{[x]_q}{\lambda}} (1+\lambda t)^{\frac{q^x [y]_q}{\lambda}} \\ &= \left(\sum_{m=0}^{\infty} ([x]_q)_{m,\lambda} \frac{t^m}{m!} \right) \left(\sum_{l=0}^{\infty} \lambda^{-l} q^{lx} \frac{[y]_q^l (\log(1+\lambda t))^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \sum_{l=0}^k ([x]_q)_{n-k,\lambda} \lambda^{k-l} q^{lx} [y]_q^l S_1(k, l) \binom{n}{k} \right) \frac{t^n}{n!}. \end{aligned} \quad (2.14)$$

From (1.7) and (2.14), we can derive the following theorem.

Theorem 2.5. *For $n \geq 0$, we have*

$$E_{n,q}(x|\lambda) = \sum_{k=0}^n \sum_{l=0}^k \binom{n}{k} ([x]_q)_{n-k,\lambda} \lambda^{k-l} q^{lx} S_1(k, l) E_{l,q}.$$

It is not difficult to show that

$$q \int_{\mathbb{Z}_p} ([x+y+1]_q)_{n,\lambda} d\mu_{-q}(y) + \int_{\mathbb{Z}_p} ([x+y]_q)_{n,\lambda} d\mu_{-q}(y) = [2]_q ([x]_q)_{n,\lambda}. \quad (2.15)$$

Thus, by Theorem 2.5 and (2.15), we obtain the following theorem.

Theorem 2.6. *For $n \geq 0$, we have*

$$qE_{n,q}(x+1|\lambda) + E_{n,q}(x|\lambda) = [2]_q ([x]_q)_{n,\lambda}.$$

Let $r \in \mathbb{N}$. Now, we recall the Carlitz's q -Euler polynomials of order r which are given by the generating function to be

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{[x_1 + \cdots + x_r + x]_q} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) = \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x) \frac{t^n}{n!}. \quad (2.16)$$

Thus, by (2.16), we get

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_r + x]_q^n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) = E_{n,q}^{(r)}(x), \quad (n \geq 0), \quad (\text{see [5]}). \quad (2.17)$$

From (2.17), we have

$$\begin{aligned} E_{n,q}^{(r)}(x) &= \frac{[2]_q^r}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lx} \left(\frac{1}{1+q^{l+1}} \right)^r \\ &= [2]_q^r \sum_{m=0}^{\infty} (-1)^m \binom{r+m-1}{m} q^m [m+x]_q^n. \end{aligned} \quad (2.18)$$

Thus, by (2.18), we get the generating function of Carlitz's q -Euler polynomials of order r which given by

$$\sum_{n=0}^{\infty} E_{n,q}^{(r)}(x) \frac{t^n}{n!} = [2]_q^r \sum_{m=0}^{\infty} (-1)^m \binom{r+m-1}{m} q^m e^{[m+x]_q t}. \quad (2.19)$$

We consider *degenerate q -Euler polynomials of order r* which are given by the generating function to be

$$\sum_{n=0}^{\infty} E_{n,q}^{(r)}(x|\lambda) \frac{t^n}{n!} = [2]_q^r \sum_{m=0}^{\infty} (-1)^m \binom{r+m-1}{m} q^m (1+\lambda t)^{[m+x]_q t}. \quad (2.20)$$

Thus, by (2.20), we have

$$\begin{aligned} &\sum_{n=0}^{\infty} E_{n,q}^{(r)}(x|\lambda) \frac{t^n}{n!} \\ &= [2]_q^r \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} (-1)^m \binom{r+m-1}{m} q^m ([m+x]_q)_{n,\lambda} \right) \frac{t^n}{n!}. \end{aligned} \quad (2.21)$$

By comparing the coefficients on the both sides of (2.21), we obtain the following theorem.

Theorem 2.7. *For $n \geq 0$, we have*

$$E_{n,q}^{(r)}(x|\lambda) = [2]_q^r \sum_{m=0}^{\infty} (-1)^m \binom{r+m-1}{m} q^m ([m+x]_q)_{n,\lambda}.$$

From (1.5) and Theorem 2.7, we note that

$$\sum_{n=0}^{\infty} E_{n,q}^{(r)}(x|\lambda) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{[x_1 + \cdots + x_r + x]_q}{\lambda}} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r). \quad (2.22)$$

Thus, by (2.22), we get

$$\begin{aligned} E_{n,q}^{(r)}(x|\lambda) &= \sum_{m=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} ([x_1 + \cdots + x_r + x]_q)_{n,\lambda} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) \\ &= \sum_{m=0}^n S_1(n, m) \lambda^{n-m} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_r + x]_q^m d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r). \end{aligned} \quad (2.23)$$

Therefore, by (2.16) and (2.23), we obtain the following theorem.

Theorem 2.8. *For $n \geq 0$, we have*

$$E_{n,q}^{(r)}(x|\lambda) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} ([x_1 + \cdots + x_r + x]_q)_{n,\lambda} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r).$$

Furthermore,

$$E_{n,q}^{(r)}(x|\lambda) = \sum_{m=0}^n S_1(n, m) \lambda^{n-m} E_{m,q}^{(r)}(x).$$

Replacing t by $\frac{1}{\lambda}(e^{\lambda t} - 1)$ in (2.22), we get

$$\begin{aligned} & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{[x_1 + \cdots + x_r + x]_q t} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) \\ &= \sum_{m=0}^{\infty} E_{m,q}^{(r)}(x|\lambda) \frac{1}{m!} \lambda^{-m} (e^{\lambda t} - 1)^m \\ &= \sum_{m=0}^{\infty} E_{m,q}^{(r)}(x|\lambda) \lambda^{n-m} S_2(n, m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \lambda^{n-m} E_{m,q}^{(r)}(x|\lambda) S_2(n, m) \right) \frac{t^n}{n!}. \end{aligned} \quad (2.24)$$

Therefore, by (2.16) and (2.24), we obtain the following theorem.

Theorem 2.9. *For $n \geq 0$, we have*

$$E_{n,q}^{(r)}(x) = \sum_{m=0}^n E_{m,q}^{(r)}(x|\lambda) \lambda^{n-m} S_2(n, m).$$

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